Massachusetts Institute of Technology

Transport Phenomena

Unit (IV) – Fluid Flow

Bernoulli Equation

Teaching Seminar

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Context & Studying Material

For a better preparation of Unit (IV), some of the **recommended literature**:

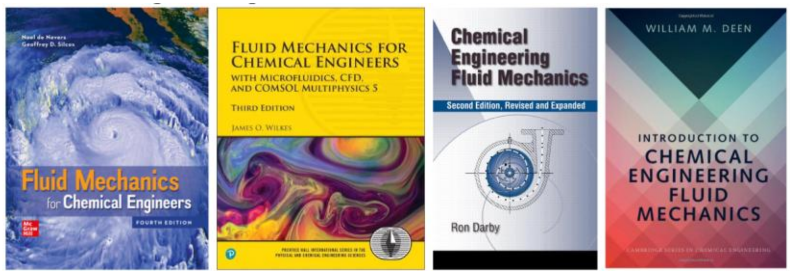
Unit (IV) – *Bernoulli's equation*

(1) ***Fluid Mechanics: Fundamentals and Applications*** (McGraw Hill) by Yunus Çengel and John Cimbala

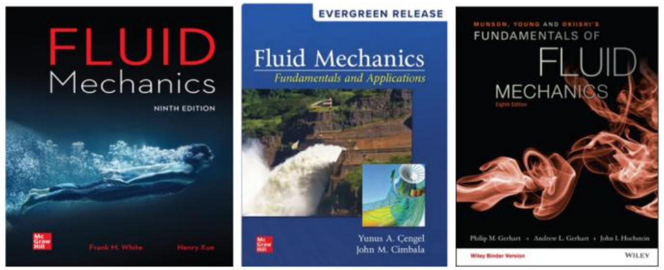
(2) ***Fluid Mechanics for Chemical Engineers*** (Pearson) by James Wilkes

(3) Fundamentals of Fluid Mechanics (Wiley) by Bruce Munson, Donald Young, and Theodore Okiishi

Chemical Engineering Fluids Textbooks



General Fluid Mechanics Textbooks



Today's Objectives

By the end of this session, YOU will be able to:

(i) ***Formulate*** Bernoulli's equation and list the variables involved

(ii) ***Apply*** Bernoulli's equation in a variety of problems, including flow *velocity* measurements and *pressure* calculations

Team Project

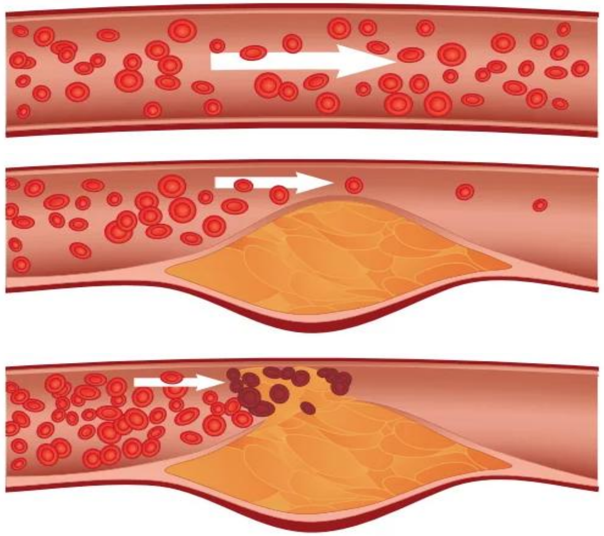
Please take a look at the handout!

Note: More information will be provided at the end of this session!

Let’s reflect on our world!

Why do airplanes fly (**air**)?

Why does **water** move faster in a squeezed hose?



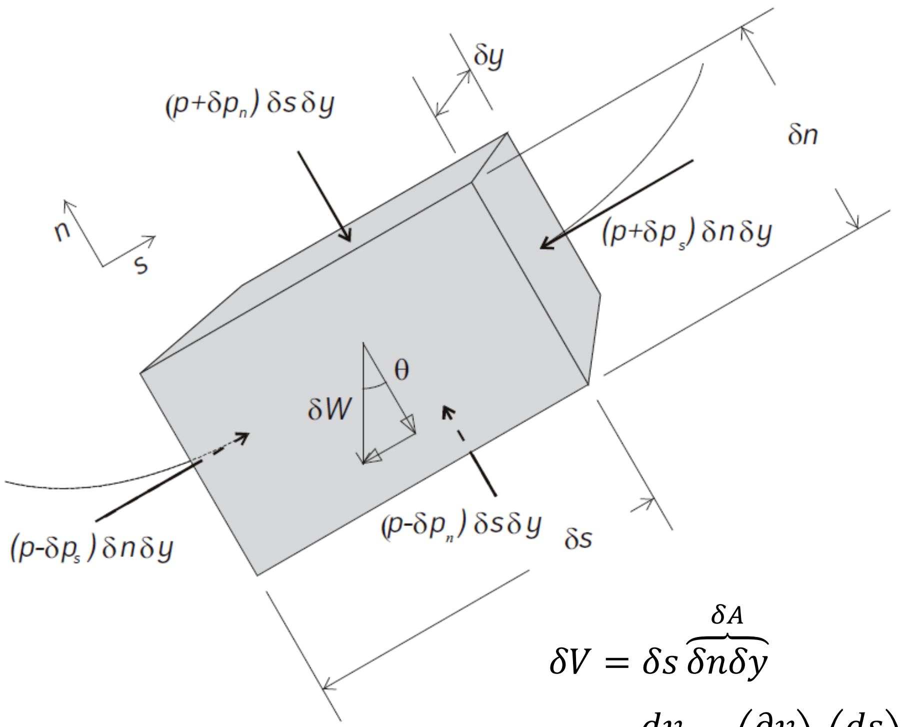
Why does an artery with a blockage cause faster **blood** flow in the constricted area?

Daniel Bernoulli

Bernoulli’s Principle!

Bernoulli’s Principle: Derivation

Considering an infinitesimal fluid element with dimensions δn and δs in the figure plane and δy in the direction normal to this plane:



$$\delta V = \delta s \overbrace{\delta n \delta y}^{\delta A}$$

$$a_s = \frac{dv}{dt} = \left(\frac{\partial v}{\partial s}\right)\underbrace{\left(\frac{ds}{dt}\right)}_{v}$$

(1) In a steady-state, applying **Newton’s second law**, along the streamline direction s

$$\sum \delta F_s = \delta m a_s = \delta m v_s \frac{\delta v_s}{\delta s} = \boldsymbol{\rho \delta V v_s \frac{\delta v_s}{\delta s}}$$

(2) Forces applied to the fluid element

$$\sum \delta F_s = \delta W_s + \delta F_{p,s}$$

- **Gravity force** along the streamline direction

$$\delta W_s = -\rho g \sin\theta\, \delta V$$

- **Pressure force** along the streamline direction

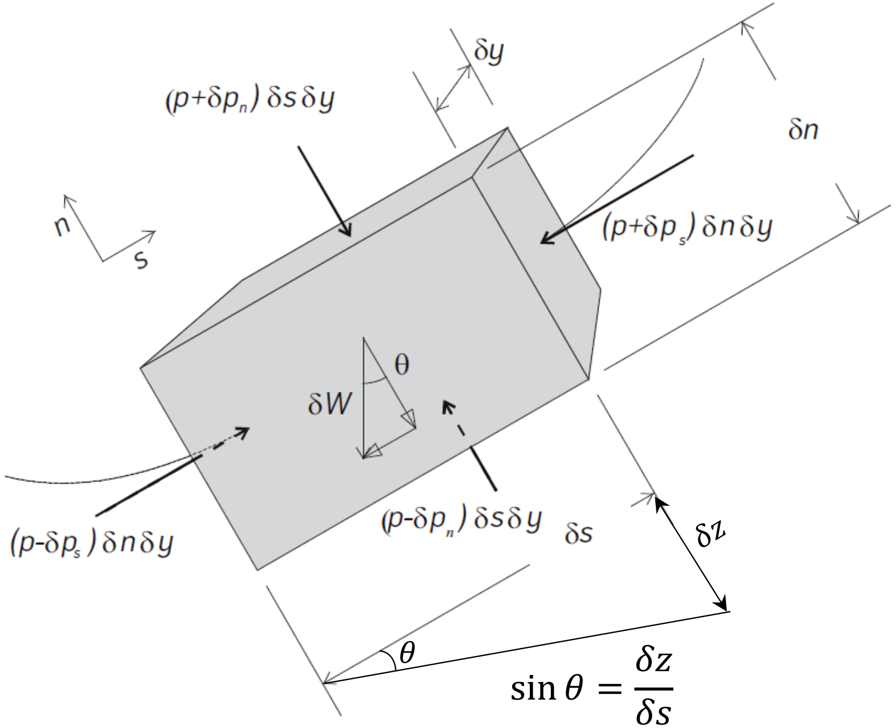
$$\delta p_s = \frac{1}{2}\left(\frac{\delta p}{\delta s}\delta s\right) \longrightarrow \delta F_{p,s} = (p - \delta p_s)\delta n \delta y - (p + \delta p_s)\delta n \delta y \Longrightarrow$$

$$\Longrightarrow\ \delta F_{p,s} = -2\delta p_s \delta n \delta y = -\frac{\partial p}{\partial s}\delta s \delta n \delta y = -\frac{\partial p}{\partial s}\delta V$$

$$\sum \delta F_s = \delta W_s + \delta F_{p,s} = \left(\boldsymbol{-\rho g \sin\theta - \frac{\partial p}{\partial s}}\right)\boldsymbol{\delta V}$$

Bernoulli's Principle: Derivation

Considering an infinitesimal fluid element with dimensions δn and δs in the figure plane and δy in the direction normal to this plane:



$$\sin\theta = \frac{\delta z}{\delta s}$$

$$dp = \frac{\partial p}{\partial s} ds + \frac{\partial p}{\partial n} dn$$

$$v_s \frac{\delta v_s}{\delta s} = \frac{1}{2}\frac{d(v^2)}{ds}$$

(3) Combining (1) + (2)

$$\sum \delta F_s = \delta m a_s = \delta m v_s \frac{\delta v_s}{\delta s} = \rho \delta V v_s \frac{\delta v_s}{\delta s}$$

$$\sum \delta F_s = \delta W_s + \delta F_{p,s} = \left(-\rho g \sin\theta - \frac{\partial p}{\partial s}\right) \delta V$$

$$\rho v_s \frac{\delta v_s}{\delta s} = \left(-\rho g \sin\theta - \frac{\partial p}{\partial s}\right) \Longrightarrow$$

$$\Longrightarrow \frac{1}{2}\rho \frac{d(v^2)}{ds} + \rho g \frac{dz}{ds} + \frac{dp}{ds} = 0$$

$$\Longrightarrow dp + \frac{1}{2}\rho d(v^2) + \rho g dz = 0$$

$$\Longrightarrow \int dp + \frac{1}{2}\rho v^2 + \rho g z = \text{const}$$

$$\Longrightarrow \boldsymbol{p + \frac{1}{2} v^2 + \rho g z = \mathbf{const}}$$

Bernoulli’s Principle: Explanation

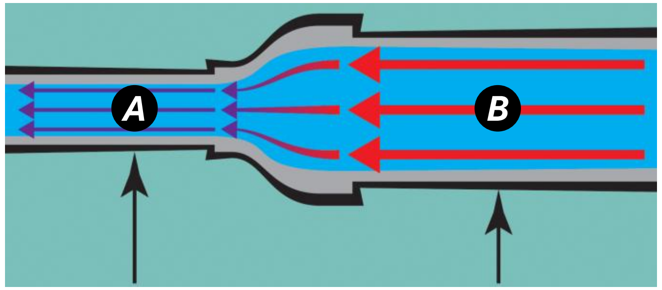
Conservation of energy in a fluid

$$\boldsymbol{P} + \frac{1}{2}\rho u^2 + \rho g h = \text{constant}$$

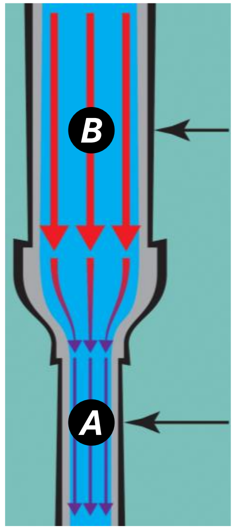
Pressure energy (Pa = kg·m^{-1}·s^{-2})

Kinetic energy per unit volume (kg·m^{-1}·s^{-2})

Potential energy per unit volume (kg·m^{-1}·s^{-2})



- $h_A = h_B$
- $u_A > u_B$
- $\boldsymbol{P_A < P_B}$



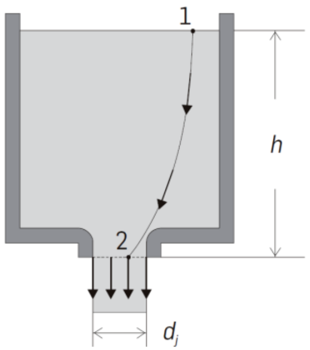
- $h_A < h_B$
- $u_A > u_B$
- $\boldsymbol{P_A = P_B}$

Bernoulli's Principle: Key points

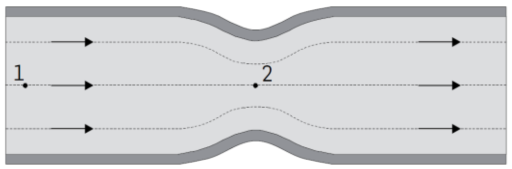
Conservation of energy in a fluid

$$P + \frac{1}{2}\rho u^2 + \rho g h = \text{constant}$$

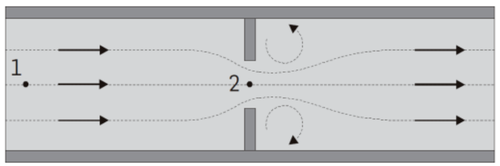
- Bernoulli's equation is an **energy equation** derived from **Newton's second law**
- Bernoulli's equation assumes **incompressible**, **inviscid**, **steady-state flow** along a streamline
- Although **gases** are not usually considered to be incompressible, Bernoulli's equation can be used when the velocity of the fluid divided by the speed of sound (Mach number) is less than 0.3
- Bernoulli's equation can be applied in different chemical engineering scenarios, such as:



Free jets

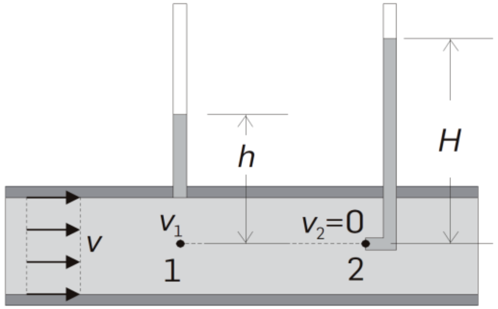


Venturi meter



Orifice meter

Pipe flow meter



Pitot tube

Velocity meter

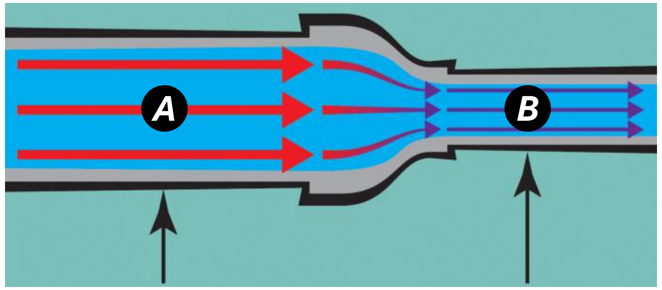
Bernoulli's Principle: Applications

Example I

Lemonade flows through a horizontal pipe where the diameter decreases. If the velocity in the wider section is 2 m·s^{-1} and the pressure is 50 kPa, what happens to the pressure when the velocity increases to 4 m·s^{-1}?

$v_A = 2\ \mathrm{m \cdot s^{-1}}$
$P_A = 50\ \mathrm{kPa}$

$v_B = 4\ \mathrm{m \cdot s^{-1}}$
$\boldsymbol{P_B} = ?$



$$\boldsymbol{P + \frac{1}{2}\rho u^2 + \rho g h = \mathrm{const}}$$

$$\boldsymbol{P_A + \frac{1}{2}\rho u_A^2 = P_B + \frac{1}{2}\rho u_B^2}$$

$$P_B = \frac{1}{2}\rho(u_A^2 - u_B^2) + P_A$$

$$P_B = 44\ \mathrm{kPa}$$

Bernoulli's Principle: Explanation

Example II

An airplane wing has air moving over the top surface at 250 m·s⁻¹, while the air below the wing moves at 200 m·s⁻¹. If the density of air is 1.225 kg·m⁻³, find the pressure difference between the bottom and top of the wing.

$v_1 = 250\ \mathrm{m \cdot s^{-1}}$
$v_2 = 200\ \mathrm{m \cdot s^{-1}}$

$\boldsymbol{P_2 - P_1} = ?$

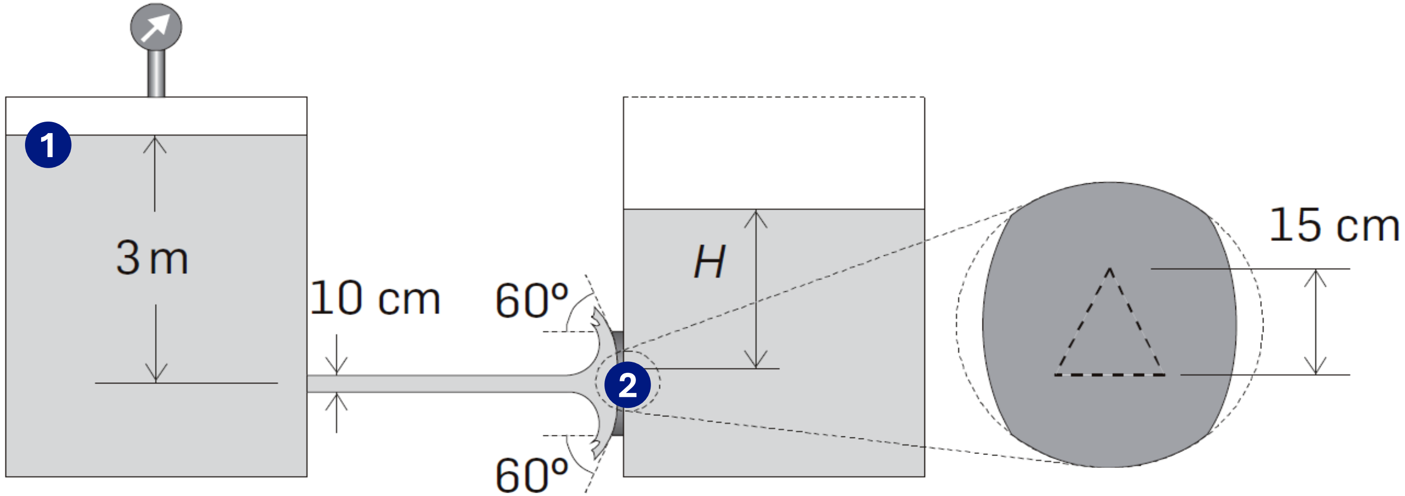
$$\boldsymbol{P_2 - P_1} = \Delta \boldsymbol{P} = \frac{\boldsymbol{1}}{\boldsymbol{2}}\boldsymbol{\rho}\left(\boldsymbol{u_1^2} - \boldsymbol{u_2^2}\right)$$

$\Delta P \simeq 13.8\ \mathrm{kPa}$

🚀 This pressure difference generates **lift**, allowing the airplane to fly!

Bernoulli's Principle: Final Problem – Think–Pair–Share

A plate covers the orifice of a reservoir as shown in the figure. The plate is held in place by a free jet of water exiting a pressurized reservoir. Determine the velocity at which the free jet hits the plate. The first reservoir is pressurized at 1.5 atmospheric pressure.



(i) Apply Bernoulli's equation between **(1)** (the surface of the pressurized reservoir and **(2)** (a point in the free jet)

$$p_1 + \frac{1}{2}\rho v_1^2 + \rho g z_1 = p_2 + \frac{1}{2}\rho v_2^2 + \rho g z_2$$

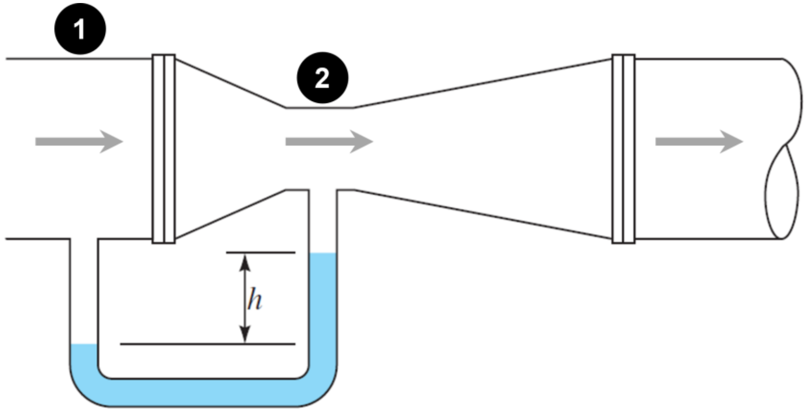
Simplifications: (1) $\begin{cases} p_1 = 1.5\, p_{\text{atm}} \\ p_2 = p_{\text{atm}} \end{cases}$ (2) $(z_1 - z_2) = h = 3\ \text{m}$ (3) $v_1 \longrightarrow 0$

Thus, $\boldsymbol{v_2} = \sqrt{\boldsymbol{2gh - \frac{2}{\rho}(p_2 - p_1)}} \approx 12.6\ \text{m}\cdot\text{s}^{-1} \longrightarrow v_{2,x} = v_2 \cos 60° = \dfrac{v_2}{2}$

Bernoulli's Principle: Demonstration

Venturi, a type of flow meter, is shown in the figure. Demonstrate that the flow rate (Q) is related to the height h read on the mercury manometer (ρ_M) according to the expression

$$Q = \frac{\pi}{4} \frac{D_2^2}{\sqrt{1-(D_2/D_1)^4}} \sqrt{\frac{2gh(\rho_{\mathrm{M}}-\rho)}{\rho}} .$$



(i) Apply Bernoulli's equation between points (1) and (2)

$$p_1 + \frac{1}{2}\rho v_1^2 + \rho g z_1 = p_2 + \frac{1}{2}\rho v_2^2 + \rho g z_2$$

Simplification: $z_1 = z_2$

$$p_1 + \frac{1}{2}\rho v_1^2 = p_2 + \frac{1}{2}\rho v_2^2$$

Bernoulli’s Principle: Demonstration

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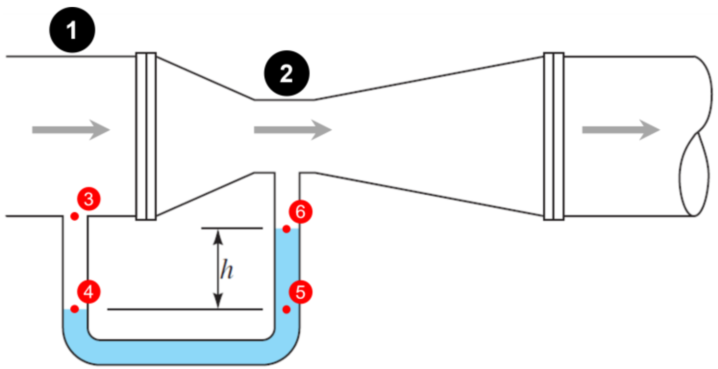

$$p_3 = p_1$$

$$p_4 = p_3 + \rho g h$$

$$p_5 = p_4$$

$$p_6 = p_5 - \rho_{\mathrm{M}} g h$$

$$p_2 = p_6$$

(ii) Obtain an expression that relates the terms p_1 and p_2 through the manometer reading

$$p_2 = p_1 + gh(\rho - \rho_{\mathrm{M}})$$

$$\frac{1}{2}\rho v_1^2 = gh(\rho - \rho_{\mathrm{M}}) + \frac{1}{2}\rho v_2^2$$

(iii) Obtain an expression that relates the terms v_1 and v_2 through the continuity equation ($\dot{m}_1 = \dot{m}_2$)

$$\rho v_1 A_1 = \rho v_2 A_2 \Longrightarrow v_1 = v_2 \frac{A_2}{A_1}$$

Bernoulli's Principle: Demonstration

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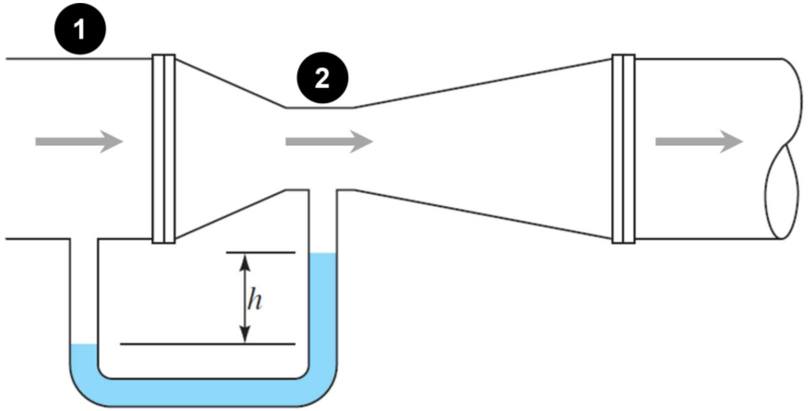


$$\begin{cases} A_1 = \frac{\pi}{4}D_1^2 \\ A_2 = \frac{\pi}{4}D_2^2 \end{cases}$$

$$\frac{1}{2}\rho v_2^2\left(\frac{D_2}{D_1}\right)^4 = gh(\rho-\rho_{\mathrm{M}}) + \frac{1}{2}\rho v_2^2 \Longrightarrow$$

$$\Longrightarrow \frac{1}{2}\rho v_2^2\left[\left(\frac{D_2}{D_1}\right)^4 - 1\right] = gh(\rho-\rho_{\mathrm{M}})$$

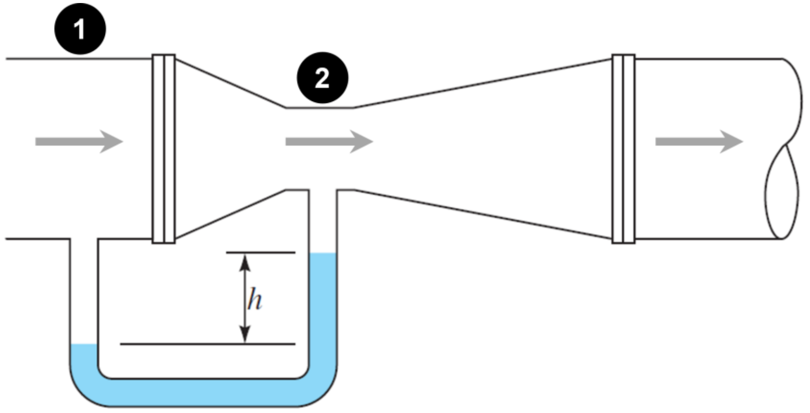
$$\Longrightarrow \frac{1}{2}\rho v_2^2\left[1-\left(\frac{D_2}{D_1}\right)^4\right] = gh(\rho_{\mathrm{M}}-\rho)$$

$$\Longrightarrow \boldsymbol{v_2} = \sqrt{\frac{\boldsymbol{2gh(\rho_{\mathrm{M}}-\rho)}}{\boldsymbol{\rho\left[1-\left(\frac{D_2}{D_1}\right)^4\right]}}}$$

Bernoulli's Principle: Demonstration

Venturi, a type of flow meter, is shown in the figure. Demonstrate that the flow rate (Q) is related to the height h read on the mercury manometer (ρ_M) according to the expression

$$Q = \frac{\pi}{4} \frac{D_2^2}{\sqrt{1-(D_2/D_1)^4}} \sqrt{\frac{2gh(\rho_{\mathrm{M}}-\rho)}{\rho}}\ .$$



(iv) Obtain an expression for the flow rate ($\boldsymbol{Q}$): $Q = v_2 A_2$

$$Q = \frac{\pi}{4} D_2^2 \sqrt{\frac{2gh(\rho_{\mathrm{M}}-\rho)}{\rho\left[1-\left(\frac{D_2}{D_1}\right)^4\right]}} \Longrightarrow$$

$$\Longrightarrow \boldsymbol{Q = \frac{\pi}{4} \frac{D_2^2}{\sqrt{1-\left(\frac{D_2}{D_1}\right)^4}} \sqrt{\frac{2gh(\rho_{\mathrm{M}}-\rho)}{\rho}}}$$

Summary

- **Bernoulli's equation** expresses the **conservation of energy** in a fluid

$$P + \frac{1}{2}\rho u^2 + \rho g h = \text{const}$$

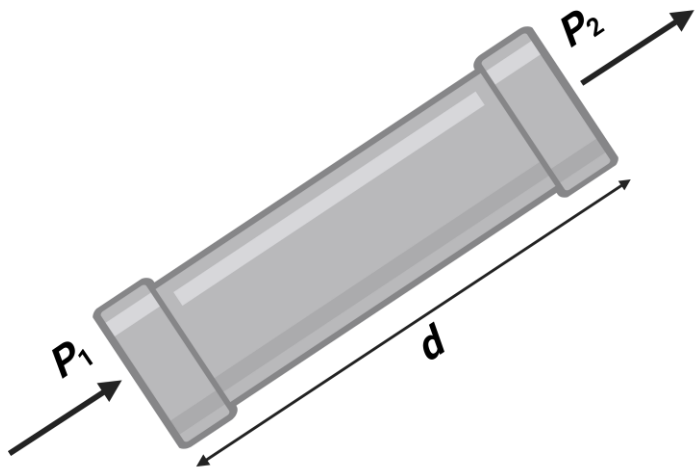
- **Bernoulli's equation** has three main terms
  - ***Pressure*** **term:** P
  - ***Kinetic energy*** **term:** $\frac{1}{2}\rho u^2$
  - ***Potential energy*** **term:** $\rho g h$

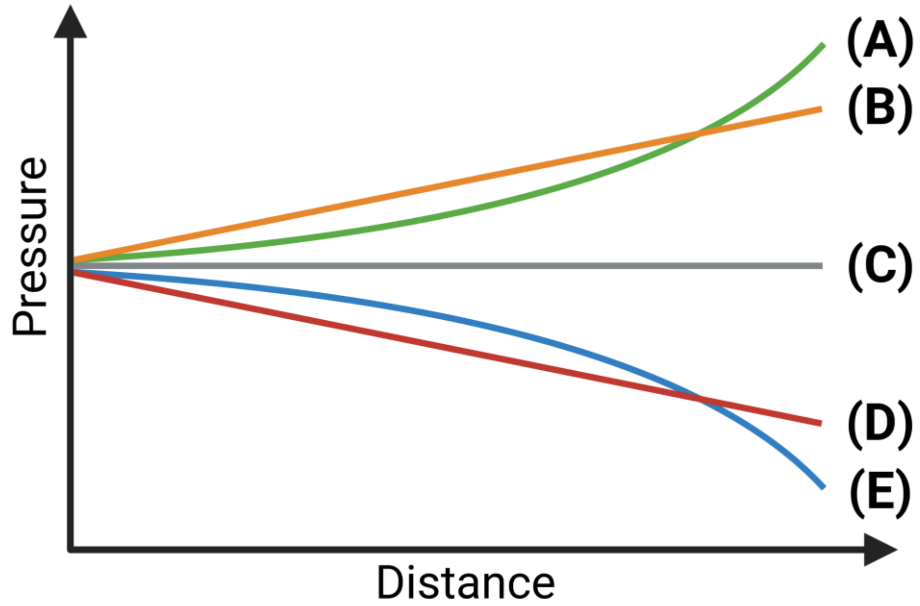
- **Bernoulli's equation** has many **real-world applications**
  - Aerodynamics
  - Medical field
  - Sports
  - Everyday life

Bernoulli’s Principle: Multiple choice questions

Question I

Water flows without friction through a constant-diameter pipe that is tilted upwards. Which curve best describes the static pressure distribution for the pipe?



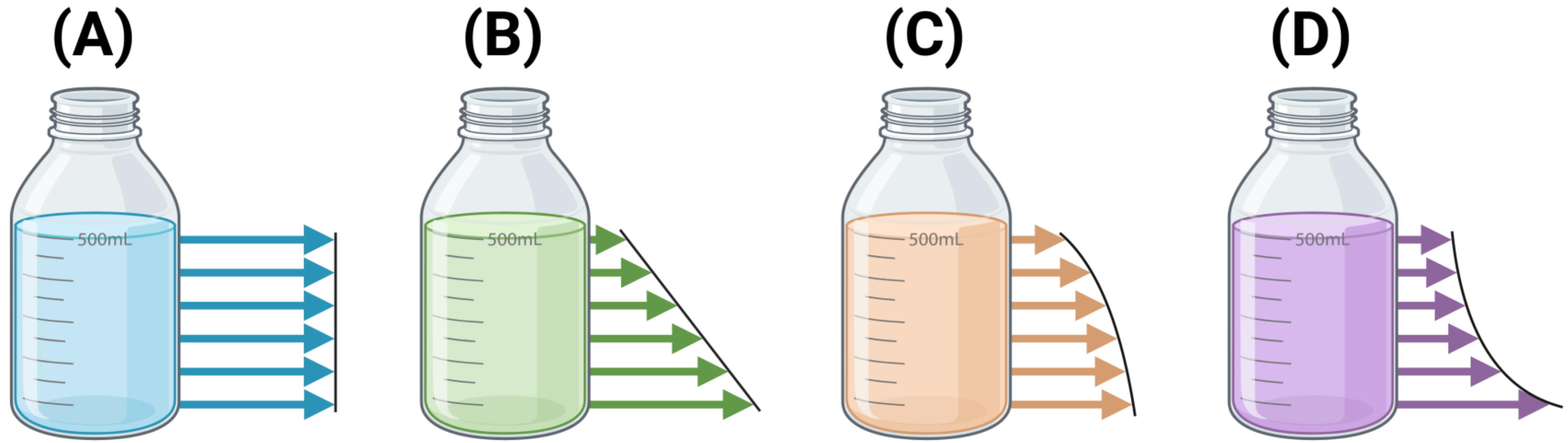


Answer: (D) – Because the area remains constant, the velocity of the fluid remains constant (no friction). The change in height (potential energy) must coincide with a **decrease in the static pressure**. This is a **linear relationship** as observed by the Bernoulli equation.

Bernoulli's Principle: Multiple choice questions

Question II

An open bottle from the lab is filled with water. Holes are made on the side in a vertical line. Which of the following diagrams represents the variation in water velocity as it leaves the holes?



Answer: **(C)** – The **velocity of the free jet** (v) is related to the **height (h) of the fluid** using the Bernoulli equation. The result is the equation for free jet velocity: $\boldsymbol{v} = \sqrt{\boldsymbol{2gh}}$. Thus, the profile would be similar to (C).

Homework

- **Choose** the *Team Project* topic

- **Reflect** on the following questions and discuss them with your colleagues:

  - **Why** do *airplanes* and *racecars* use aerodynamic shapes?
  - **How** do *firefighters* use water hoses with nozzles?
  - **Why** do *baseball pitchers* use curveballs to control ball movement?

Massachusetts Institute of Technology

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Teaching Seminar

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Toronto, 4th of March 2026